**Thermal Properties**

Now we’ll examine the thermodynamic properties of these spin systems coupled via the exchange interaction.

**Ising Model**

Going to jump straight to the Ising model we’ve been using, but if you want a little more background and context, can check out the Stat Mech folder.



To streamline stuff, we’ll use units where ℏ = 1, and kB = 1. Also going to use **S** = **σ**/2. So can write:



And gonna absorb the 1/4 into the J. And absorb the (1/2)gγBf into a single term h – my favorite letter.



So now J and h both have units of energy. Okay so finally now we have (z directions implicit):



Note that we *do* know the eigenstates and energies of *this* system – as it comprises a bunch of purely independent spins, and so the eigenstates are just |ψ> = |σ1 = ±1,σ2 = ±1,σ3 = ±,…,σN = ±1>, and the eigenvalues are of course whatever you get when you plug this into HIsing. So we *could* exactly calculate the partition function.



This is practical in 1D, really really really difficult in 2D, and impossible in 3D? Let’s do 1D. We’ll presume periodic boundary conditions such that for our N spins σ1 = σN+1. Then we start by writing our Z in a more symmetric form,



Then let’s write Z as:



where we recognize the factor P can be represented as a 2×2 matrix, with row index given by σi = ±1, and column index by σi+1 = ±1. Or in other words,



where σ and σ´ can both be ±1. So then our Z can be written,



From Quantum Mechanics, or Linear Algebra, we’ll recognize this as the Trace of N, where is the matrix with the elements Pσσ´. So we have:



This can be evaluated by finding the eigenvector decomposition of . Let = UΛU-1, where Λ is the matrix of eigenvalues and U the matrix of eigenvectors. Then,



Using the cyclic property of the Tr, we can say,



where λ1,2 are the two eigenvalues of . What are these? Gotta solve,



This works out to,



Can simplify these to:



Now both eigenvalues are real, as guaranteed by the fact that is symmetric. And λ+ > λ-. This means that (λ+)N >> (λ-)N. And so the contribution of λ+ to Z will be much greater than the contribution of λ-. Thus, we may say,



and the Free energy per unit spin is F = F/N:



which is:



The average magnetic moment of each spin will be m = -∂F/∂Bf, but I’m going to calculate instead m = -∂F/∂h (recall h = 1/2·gγBf).



So we have:



From this, we can see that in the low T limit, there is no finite T transition to m → 1. Thus no spontaneous magnetization occurs, and we do not have a phase transition in 1D. But as T → 0, we do get m → 1,



So at T = 0, the spin is completely aligned with the field, as we expect. It will be of interest later to examine the behavior of F near the ‘critical point’, T = 0. So consider βF(K,j) where K = βJ and j = βh.



and in the large K (small T), small j limit we have:

